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$L(1,1)-$ LABELING OF DIRECT PRODUCT OF CYCLES

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Abstract. An $L(1,1)$-labeling of a graph G is an assignment of labels from $\{0, 1 \cdots, k\}$ to the vertices of G such that two vertices that are adjacent or have a common neighbor receive distinct labels. The λ_1^1- number, $\lambda_1^1(G)$ of G is the minimum value k such that G admits an $L(1,1)$ labeling. We establish the λ_1^1- numbers for direct product of cycles $C_m \times C_n$ for all positive $m, n \geq 3$, where both m, n are even or when one of them is even and the other odd.

1. Introduction

The $L(h,k)$-labeling problem (or (h,k)-coloring problem) is that of vertex labeling of an undirected graph G with non-negative integers such that for every $u, v \in V(G)$, $uv \in E(G)$, $|l(u) - l(v)| \geq h$ and for all $u, v \in V(G), d(u,v) = 2$, $|l(u) - l(v)| \geq k$. The difference between the largest label and the smallest label assigned is called the span. The aim of $L(h,k)-$labeling is to obtain the smallest non negative integer $\lambda_h^k(G)$, such that there exists an $L(h,k)$-labeling of G with no label on $V(G)$ greater than $\lambda_h^k(G)$.

Motivated by Hales' 1980 paper [8], which provided a new model for frequency assignment problems as a graph coloring problem, Griggs and Yeh [7] formulated the $L(2,1)$ problem to model the channel assignment problem. The general notion of $L(h,k)$- labeling was first presented by Georges and Mauro [6] in 1995. The topic has since then been an object of extensive research for various graphs. Calamonerri's survey paper [4] contains known results on $L(h,k)$-labeling of graphs.

$L(1,1)$-labeling (or strong labeling condition) of a graph is a labeling of G such that vertices with a common neighbor are assigned distinct labels. The usual labeling (or proper vertex coloring) condition is that adjacent vertices have different colors, but for $L(1,1)$, also all neighbors of any vertex are colored differently. This is equivalent to a proper vertex-coloring of the square of a graph G. Note that a proper k- coloring of a graph is a mapping $\alpha : V(G) \to \{1, \cdots, k\}$ such that for all $uv \in E(G)$ $\alpha(u) \neq \alpha(v)$ and the square G^2 of G has vertex $V(G)$ with an edge between two vertices which are adjacent in G or have a common neighbor in G. The chromatic number $\chi(G)$ of G is the smallest k for which G admits a k-coloring. Therefore, $\chi(G^2) = \lambda_1^1(G) + 1$ for a graph G.

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Labeling of graph powers is often motivated by applications in frequency assignment and has attracted much attention [See for example, [1]]. $L(1,1)$-labeling has applications in computing approximation to sparse Hessian matrices, design of collision-free multi-hop channel access protocols in radio networks segmentation problem for files in a network and drawings of graphs in the plane [3, 13, 15, 16] to mention a few.

For graphs G and H, the direct product $G \times H$ have vertex set $V(G) \times V(H)$ where two vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if $(x_1, y_1) \in E(G)$ and $(x_2, y_2) \in E(H)$. This product is one of the most important graph products with potential applications in engineering, computer science and related disciplines. [11]. The $L(h,k)$- labeling of direct product of graphs was investigated in [2, 5, 9, 12, 14, 18, 19, 20].

In particular, Jha et al [12] gave upper bounds for λ_1^k-labeling of the multiple direct product and cartesian product of cycles with some conditions on k and the length of the cycles. They also presented some cases where we have exact values. In addition by using backtracking algorithm, they computed $\lambda_1^d(C_m \times C_n)$ for $2 \leq d \leq 4$ and $4 \leq m,n \leq 10$. Since every $L(2,1)$-labeling is an $L(1,1)$-labeling, then $\lambda_1^1(G) \leq \lambda_1^2(G)$. Therefore, their results for $d = 2$ provided upper bounds for $L(1,1)$-labeling of $C_m \times C_n$ for $4 \leq m,n \leq 10$. The only result for λ_1^1-labeling for direct product of two cycles in the paper is that if $m, n \equiv 0 mod 5$ then $\lambda_1^1(C_m \times C_n) = 4$.

In this paper, we solve the $L(1,1)$-labeling problem for direct product of cycles C_m, C_n, $m, n \geq 3$, except for $m \in \{16, 18, 22, 26, 32, 36, 46\}$, $n \in \{14, 16, 18, 26, 28, 34\}$ and for these outstanding cases we conjecture that $\lambda_1^1(C_m \times C_n) = 5$.

The paper is organized as follows: We give some preliminaries in Section 2 and obtain the λ_1^1 labeling numbers for $C_m \times C_n$ for $m \geq 3$ and $n = 4$ and 6 and some of their multiples in Section 3. Section 4 deals with labeling of direct product of bigger cycles.

2. Preliminaries

Let G be a finite simple undirected graph with at least two vertices. For subgraph $V' \subseteq V(G)$, we denote by $L(V')$ the set of $L(1,1)-$labeling on V' and for a non-negative integer, say, k, we take $[k(\epsilon)]$ as the set of even integers and zero in $[k]$ while $[k(o)]$ is the set of odd integers in $[k]$. Suppose further that $v \in V(G)$, we denote d_v as the degree of v.

The following results, remarks and definitions are needed in the work.

Theorem 2.1. [10] *Graph $G \times H$ is connected if and only if G and H are connected and at least one of G and H is non-bipartite.*

Remark 2.2. (i) *Let $G = C_m \times C_n$, where m, n are even positive integers. Then, $G = G_1 \cup G_2$, where G_1 and G_2 are the connected components of $C_m \times C_n$, where*

$$\{V(G_1) = u_i v_j : i \in [(m-1)(\epsilon)], j \in [(n-1)(\epsilon)] \text{ or }$$
$$i \in [(m-1)(o)]; j \in [(n-1)(0)]\} \textit{ and}$$
$$\{V(G_2) = u_i v_j : i \in [(m-1)(\epsilon)], j \in [(n-1)(o)] \text{ or }$$

$$i \in [(m-1)(o)]; j \in [(n-1)(\epsilon)]\} .$$

Note that G_1 *and* G_2 *are isomorphic and it is demonstrated in the graph* $C_4 \times C_6$ *below*

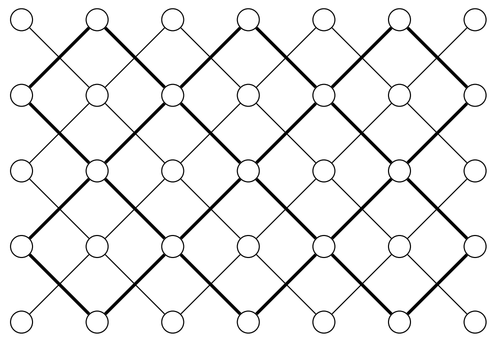

Fig. 1: The components of $C_4 \times C_6$

(ii) *Suppose* $G = C_m \times C_n$ *such that* $G = G' \cup G''$, *where* G', G'' *are components of* G, *then,* $\lambda_1^1(G) = max\{\lambda_1^1(G'), \lambda_1^1(G'')\}$.

(iii) *Let* $G = C_m \times C_n$, *where* m *is even and* n *odd positive integers. Then,* $G \equiv G_1$, *where* G_1 *is any of the two connected components of* $C_m \times C_{2n}$.

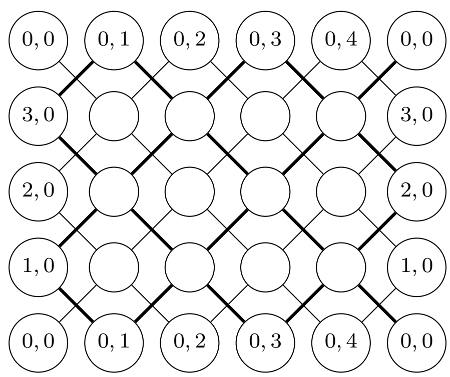

Fig. 2: $C_4 \times C_5$ *is isomorphic to a component of* $C_4 \times 10$

Let P_m be a path of length $m-1$. The following results are from [2]:

Corollary 2.3. *For* $m \geq 3$, $\lambda_1^1(P_m \times C_6) = 5$

Lemma 2.4. *For* $m \geq 5$, $n \geq 9$, $n \not\equiv 0 \mod 5$ $\lambda_1^1(P_m \times C_n) \geq 5$

A useful lower bound on $L(1,1)$-labeling for any graph G is contained in the following Lemma:

Lemma 2.5. [6] *If* G *is a graph with maximum degree* $\triangle$, *and* G *includes a vertex with* $\triangle$ *neighbors, each of which is of degree* $\triangle$, *then* $\lambda_1^1(G) \geq \triangle$

From the lemma, we have for $m, n \geq 3$ $\lambda_1^1(C_m \times C_n) = 4$

3. Labeling of $C_m \times C_n, n = 4, 6$

In this section, we investigate the $\lambda-$ numbers of graph product $C_m \times C_4$ and $C_m \times C_6$, where $m \geq 3$.

Let G' be the connected component of the product graph under consideration.

Lemma 3.1. *For $m \geq 4$, and even, $\lambda_1^1(C_m \times C_4) \geq 5$.*

Proof. Let $G' \subset C_m \times C_4$, where $m \geq 4$ and even. Suppose $V_i, V_{i+1}, V_{i+2} \subset V(G')$. Let G_1' be the subgraph of G' induced by V_{i+j}, for all $j \in [2]$. Then, $V(G_1')$=$\{u_iv_0, u_iv_2, u_{i+1}v_1, u_{i+1}v_2, u_{i+2}v_0, u_{i+2}v_2\}$. Now it is clear that the diameter of G_1' is 2. Thus for every pair $v_1, v_2 \in V(G_1')$, $d(v_1, v_2) \leq 2$. Thus, $l(v_1) \neq l(v_2)$ for all $v_1, v_2 \in V(G_1')$. Now, $|V(G_1')| = 6$. Therefore $\lambda_1^1(C_m \times C_4) \geq 5$. □

Remark 3.2. *Note that if $G' \subset C_m \times C_4$, $m \geq 4$ with m even and $v_i \in V_i$, for some i, $V_i \subset V(G')$, such that $l(v_i) = \alpha_i \in [m]$, with $\lambda_1^1(G') = m$, then $\alpha_1 \notin L\{V_{i-2}V_{i-1}V_{i+1}V_{i+2}\}$.*

Theorem 3.3. $\lambda_1^1(C_4 \times C_4) = 7$

Proof. Let $G' \subset C_4 \times C_4$ and $V_i \subset V(G')$ for each $i \in [3]$. Clearly, $V(G') = \cup_{i=0}^3 V_i$. Let G_1' be a subgraph of G' such that G_1' is induced by V_0, V_1, V_2. By the proof of Lemma 3.1, $|V(G_1')| = 6$ and suppose $\alpha_k', \alpha_k'', \in L(V_3)$, then by remark 3.2,$\alpha_k', \alpha_k'', \notin L(V(G_1'))$. Thus there exists $\alpha_k', \alpha_k'', \notin [5]$ such that $\{\alpha_k', \alpha_k''\} = L(V_3)$, and $\alpha_k' \neq \alpha_k''$ since $d(v_3', v_3'') = 2$ for $v_3', v_3'' \in V_3$. Thus, $|L(\cup_{i=0}^3 V_i)|$=$|L(V(G'))| = 6 + 2$. Therefore, $\lambda_1^1(C_4 \times C_4) = \lambda_1^1(G') = 7$. □

Next we present the necessary and sufficient condition under which $\lambda_1^1(C_m \times C_4)$ is 5.

Theorem 3.4. *For $m \geq 4, m$ even, $\lambda_1^1(C_m \times C_4) = 5$ if and only if $m \equiv 0$ mod 6.*

Proof. Let $m = 6n,\ n \in \mathbb{N}$. By Lemma 3.1, $\lambda_1^1(G) \geq 5$. Therefore, $\lambda_1^1(G') \geq 5$, where $G' \subset C_m \times C_4$. Let G'' be the connected component of $C_6 \times C_4$. By Corollary 3.2, $L(V_0) \cap L(V_1) = \emptyset$, $L(V_1) \cap L(V_2) = \emptyset$, $L(V_2) \cap L(V_0) = \emptyset$. Now, set $L(V_0) = L(V_3)$, $L(V_1) = L(V_4)$,$L(V_2) = L(V_5)$. But $L(V_5) \cap L(V_0) = \emptyset$, $L(V_5) \cap L(V_1) = \emptyset$. Thus, $\lambda_1^1(G'') \leq 5$ and $\lambda_1^1(C_6 \times C_4) = 5$. Thus by re-occurrence along C_n and C_m, $m = 0$ mod 6 implies $\lambda_1^1(G) = 5$.

Conversely, suppose $\lambda_1^1(G) = 5$. Let G' be a connected component of $G = C_m \times C_4,\ m \geq 4, m$ even. Then, $\lambda_1^1(G') = 5$. Now, assume that $m \not\equiv 0 \mod 6$, then $m = 6n' + 2$ or $m = 6n' + 4$ where $n' \in \mathbb{N} \cup 0$. For $n' = 0$, $G = C_4 \times C_4$, for which $\lambda_1^1(G') = 7$ by Theorem 3.3.

Case i: For $m = 6n' + 2,\ n' \in \mathbb{N}$, let V_0, V_1, V_2 be subsets of $V(G')$. By Corollary 3.2, $L(V_0) \cap L(V_1) = \emptyset$, $L(V_1) \cap L(V_2) = \emptyset$ and $L(V_2) \cap L(V_0) = \emptyset$. Now let G_1' be the subgraph of G' induced by V_0, V_1, V_2. Since $L(V_3) \cap L(V_2) = \emptyset$ for $V_3 \subset V(G')$ and $\lambda_1^1(G') = 5$, then $L(V_3) = L(V_0)$. Let $V_4 \subset V(G')$. Then $L(V_4) \cap L(V_3) = \emptyset$ and $L(V_4) \cap L(V_2) = \emptyset$. Thus $L(V_4) = L(V_1)$. Let $V_5 \subset V(G')$. Then $L(V_5) \cap L(V_4) = \emptyset$ and $L(V_5) \cap L(V_3) = \emptyset$ and therefore $L(V_4) = L(V_1)$. The scheme continues in such a

way that $L(V_i) = L(V_i + 3)$ for all $i \in [m-1]$, that is $L(V_0) = L(V_3) = L(V_6) = \cdots = L(V_{6n'})$, $L(V_1) = L(V_4) = L(V_7) = \cdots = L(V_{6n'+1})$, $L(V_2) = L(V_5) = L(V_8) = \cdots = L(V_0)$. Now, for all $v_a \in V_0$ and $v_b \in V_{6n'}$, $d(v_a, v_b) = 2$. For all $v_c \in V_1$, $v_d \in V_{6n'+1}$, $d(v_c, v_d) = 2$ and finally, $L(V_0) \cap L(V_2) = \emptyset$. Thus a contradiction.
Case ii: For $m = 6n' + 4$, $n' \in \mathbb{N}$, similar argument as in $m = 6n + 2$ applies. Thus, $\lambda_1^1(C_m \times C_4) = 5$ if and only $m \equiv 0 \mod 6$ □

Corollary 3.5. *Let $m \equiv 0$ mod 6 and $n \equiv 0$ mod 4. Then,* $\lambda_1^1(C_m \times C_n) = 5$

Proof. The claim follows from Theorem 3.4 and the re-occurrence of the optimal labeling of $C_{6n'} \times C_4$, $n' \in \mathbb{N}$. □

Corollary 3.6. *For all $m \not\equiv 0 \mod 6$, $\lambda_1^1(C_m \times C_4) \geq 6$.*

Theorem 3.7. *For $C_8 \times C_4$, $\lambda_1^1(C_8 \times C_4) = 7$*

Proof. Suppose G' is a connected component of $C_8 \times C_4$ and suppose that $\lambda_1^1(G') = 6$. By Corollary 3.2, $|L\{V_0, V_1, V_2\}| = 6$. Likewise, for all $\alpha_k \in L(V_0)$, $\alpha_k \notin L(V_7)$ and $\alpha_k \notin L(V_6)$. Also, for all $\alpha_j \in L(V_1)$, $\alpha_j \notin L(V_7)$. Suppose $L(V_2) = L(V_7)$, then by Corollary 3.2, if $\alpha_a, \alpha_b \in L(V_2)$, then $\alpha_a, \alpha_b \notin L\{V_3, V_4, V_5, V_6\}$. Since $\lambda_1^1(G') = 6$, then there exists only five members of [6] that labels V_3, V_4, V_5, V_6. However, this contradicts Lemma 3.1. Thus, $L(V_2) \neq L(V_7)$. Now suppose one of $\alpha_a, \alpha_b \in L(V_2)$, say α_a, labels some vertex $v_1 \in L(V_7)$, then there exists some $\alpha'_a \in [6]$ such that $\alpha'_a \notin L\{V_0, V_1, V_2\}$ such that $\alpha'_a = l(v_2) \in V_7$, with $v_1 \neq V_2$. Now let $\alpha_c, \alpha_d \in L(V_0)$. Suppose $L(V_3) = L(V_0)$. Then by Corollary 3.2, $\alpha_a, \alpha_b \notin L(V_4, V_5, V_6)$, which contradicts Lemma 3.1 since $|L(V_4, V_5, V_6)| = 6$ and $[6]\backslash 2 = 5$. Then, $\alpha'_a \in L(V_3)$ and also also one of $\alpha_a, \alpha'_b \in L(V_3)$. Further, by Corollary 3.2, $\alpha_a, \alpha'_b \notin L(V_4, V_5, V_6)$. Thus, $\lambda_1^1(G') \geq 7$. Conversely, $\lambda_1^1(G') \leq 7$ follows directly from re-occurrence of the labeling of $C_4 \times C_4$. Thus, $\lambda_1^1(G') = \lambda_1^1(C_8 \times C_4) \geq 7$. □

The next result focuses on the λ_1^1-number of $C_m \times C_4$, for $m \geq 9$. Theorem 3.6 have already established the lower bound for λ_1^1-number of $C_m \times C_4$ to be 6 if m is not a multiple of 6. So we only need to label $C_{10} \times C_4$ with [6] such that it combines perfectly with the labeling of $C_6 \times C_4$ with [5] to establish general bound for all cases except when $m = 14$ which is dealt with separately.

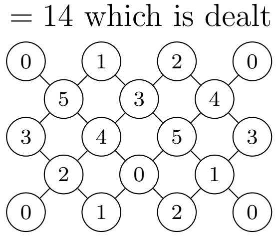

Fig. 3: $5 - L(1,1)$- labeling of $C_4 \times C_6$

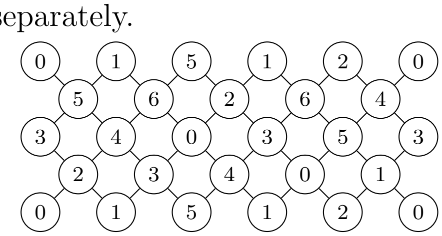

Fig. 4: $6 - L(1,1)$- labeling of $C_4 \times C_{10}$

Theorem 3.8. *Let $m', m'' \in \mathbb{N} \cup 0$, with $10m' + 6n''$ not a multiple of 6. Then* $\lambda_1^1(C_{10m'+6m''} \times C_4) = 6$.

Proof. By Corollary 3.6, $\lambda_1^1(C_m \times C_n) \geq 6$ for all m not multiple of 6. The claim follows required combinations of Figures 3 and 4 above which shows that $\lambda_1^1(C_{10m'+6m''} \times C_4) \leq 6$ for $10m' + 6m''$ not a multiple of 6. □

Clearly, every even number $m \geq 10$, $m \neq 14$ can be obtained from $10m' + 6m''$ defined above. Therefore, we can conclude that for all $m \geq 9$, $\lambda_1^1(C_m \times C_n) = 6$ for all m that is not a multiple of 6 if we can establish that the λ_1^1-number of $C_{14} \times C_4$ is 6. We show this in the next result.

Theorem 3.9. $\lambda_1^1(C_{14} \times C_4) = 6$

Proof. $\lambda_1^1(C_{14} \times C_4) = 6$ □

We have now completely determined the λ_1^1-numbers of $(C_m \times C_4)$ for all $m \geq 3$. In what follows, we investigate the values of $\lambda_1^1(C_m \times C_6)$.

Proposition 3.10. *Let G' be a connected component of $C_m \times C_6$, $m \in \mathbb{N}, m \geq 3$. Let $V_i \subseteq V(G')$, $i \in [m-1]$. Then,*
(i) $\lambda_1^1(C_m \times C_6) \geq 5$.
(ii) Given $v_a, v_b \in V_i$, $d(v_a, v_b) \leq 2$.
(iii) For all $V_i \subseteq V(G')$, $|L(V_i)| = 3$
(iv) suppose $\alpha_k \in L(V_i)$, then $\alpha_k \notin L(V_{i+2})$.

Proof. The proof of the claims above are as follows:
(i) $C_m \times C_6$ contains $P_m \times C_6$. Now from Corollary 2.3, $\lambda_1^1(P_m \times C_6) = 5$. Therefore $\lambda_1^1(C_m \times C_6) \geq 5$.
(ii) Let $V_i \subseteq V(G')$. $V_i = \{u_iv_j, u_iv_{j+2}, u_iv_{j+4}\}$, where $j \in \{0, 1\}$. Now, since C_m is a cycle,then $d(u_iv_j, u_iv_{j+4}) = 2$. Clearly, $d(u_iv_j, u_iv_{j+2}) = 2$, $d(u_iv_{j+2}, u_iv_{j+4}) = 2$ and thus the claim.
(iii) This is quite obvious.
(iv) It is obvious that for all $u_iv_j \in V_i$ and $u_{i+2}v_k \in V_{i+2}$, $d(u_iv_j, u_{i+2}v_k) = 2$. Therefore, $L(V_i) \cap L(V_{i+2}) = \emptyset$.
□

The next result describes a property of $L(1,1)$-labeling of $C_m \times C_6$

Lemma 3.11. *Let $\alpha_k \in L(V_i)$, $i \in [m-1]$, $V_i \subseteq V(G')$, then α_k labels some vertex $v_{i+1} \in V_{i+1}$. In other words, $L(V_i)$ labels V_{i+1}.*

Proof. Let $\{u_iv_j, u_iv_{j+2}, u_iv_{j+4}\} = V_i$ and $\{u_{i+1}v_{j+1}, u_{i+1}v_{j+3}, u_{i+1}v_{j+5}\} = V_{i+1}$ Clearly $d(u_iv_j, u_{i+1}v_{j+3}) = d(u_iv_{j+2}, u_{i+1}v_{j+5}) = d(u_iv_{j+4}, u_{i+1}v_{j+1}) = 3$. Therefore, suppose $\alpha_k = l(v_i)$, for some $v_i \in V_i$, then, there exists some unique $v_{i+k} \in V_{i+1}$ such that $l(v_i) = l(v_{i+k})$,with $|(i - (i+k))| = 3$. (The uniqueness of v_{i+k} results from Proposition 3.10(b).) □

Corollary 3.12. *If $L(V_i) = L(V_{i+1})$, then, $L(V_{i+2}) \cap L(V_i) = \emptyset$ and $L(V_i) \cap L(V_{i+3}) = \emptyset$.*

It is obvious from Proposition 3.10(d).

Corollary 3.13. $\lambda_1^1(C_m \times C_6) = 5$ *if and only if* $m \equiv 0$ *mod* 4

Proof. Let $m \equiv 0 \mod 4$. For $m = 4$, clearly $\lambda_1^1(C_4 \times C_6) = 5$, which is obtained from Corollary 3.6. Now in the case of the general $m \equiv 0 \mod 4$, by re-occurrence of the labeling of $C_4 \times C_6$ along C_m, it follows that $\lambda_1^1(C_m \times C_6) = 5$. Conversely, suppose that $\lambda_1^1(C_m \times C_6) = 5$. We show that $m \equiv 0 \mod 4$. Let $L(V_i) = \{\alpha_i, \alpha_j, \alpha_k\}$. By Proposition 3.10 (c),$\alpha_i \neq \alpha_j \neq \alpha_k \neq \alpha_i$, that is, $|L(V_i)| = 3$. Suppose $L(V_0) = L(V_1)$ by Lemma 3.11, then by Corollary 3.12, $L(V_2) \cap L(V_1) = \emptyset$ and $L(V_3) \cap L(V_1) = \emptyset$. Since $\lambda_1^1(C_m \times C_6) = 5$, Then $L(V_2) = L(V_3) = [5]\backslash L(V_0)$. This scheme continues such that $L(V_0) = L(V_4) = L(V_5)$; $L(V_2) = L(V_6) = L(V_7) \cdots = L(V_0) = L(V_{m-4}) = L(V_{m-3})$; and $L(V_0) = L(V_4) = L(V_8) = \cdots = L(V_{4(n)})$, $n \in \mathbb{N}$, where $4n = (m-1) + 1 = m$ since C_m is a cycle. Thus $m \equiv 0 \mod 4$. $\square$

The implication of the last result is that the lower bound for the λ_1^1-number of graph product $C_m \times C_6$, $m \geq 3$ is 6 except for when $m \equiv 0 \mod 4$, in which case the optimal λ_1^1-number reduces by 1.

Now we consider particular cases where the lower bound is strictly greater than 6.

Theorem 3.14. $\lambda_1^1(C_6 \times C_6) = 8$

Proof. Suppose that $\lambda_1^1(C_6 \times C_6) = 7$. Let $\{V_i\} \subseteq V(G')$, for all $i \in [5]$. By proposition 3.10 (d), $L(V_0) \cap L(V_2) = \emptyset$; $L(V_0) \cap L(V_4) = \emptyset$ and $L(V_4) \cap L(V_2) = \emptyset$. Now, $L(V_2) \subseteq [7]\backslash L(V_0)$ and $L(V_4) \subseteq [7]\backslash L(V_0)$. Note that $|[7]\backslash L(V_0)| = 5$. Now set $[7]\backslash L(V_0) = [A']$. $L(V_4) \subseteq [A']\backslash L(V_2)$ since $L(V_4) \cap L(V_2) = \emptyset$. Now, $|[A']\backslash L(V_2)| = 2$. However, by Proposition 3.10 (c), $|L(V_4)| = 3$. Therefore a contradiction and hence $\lambda(C_6 \times C_6) \geq 8$. The labeling in Figure 6 confirms that $\lambda_1^1(C_6 \times C_6) \leq 8$, and thus, $\lambda_1^1(C_6 \times C_6) = 8$.

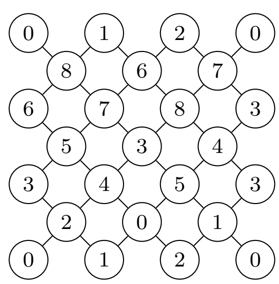

Fig. 6: $8 - L(1,1)$- labeling of $C_6 \times C_6$

$\square$

Theorem 3.15. $\lambda_1^1(C_{10} \times C_6) = 7$

Proof. Let G' be a connected component of $C_{10} \times C_6$. and suppose that $\lambda_1^1(G') = 6$. Let $V_0 \subset V(G')$ such that $L(V_0) \subset [6]$. By Proposition 3.10 (c), $L(V_0) \cap L(V_2) = \emptyset$. Therefore $L(V_0) \subset [6]\backslash L(V_0)$, where $|L(V_0)| = 3$. Thus, $|[6]\backslash L(V_0)| = 4$. Now, for all $v_0 \in V_0$ and $v_8 \in V_8$, $V_0, V_8 \subset V(G')$, $d(v_0, v_8) = 2$, since C_{10} is a cycle of length 10. Therefore, $L(V_8) \subset [6]\backslash L(V_0)$. Now, suppose that, $L(V_8) = L(V_2)$, by Proposition 3.10, then there exists $\alpha_k \in [6]$ such that $\alpha_k \notin L(V_0)$, and$\alpha_k \in L(V_2)$. Thus $L(V_4) \subset L(V_0) \cup \alpha_k$ and $L(V_6) \subset L(V_0) \cup \alpha_k$. Now, $|L(V_0) \cup \alpha_k| = 4$. By

Proposition 3.10, (d), $L(V_6) \cap L(V_4) = \emptyset$. Thus $|L(V_6) \cup L(V_4)| = 6$, which is a contradiction.

Now, suppose $L(V_8) \neq L(V_0)$ then it is not difficult to see that there exists $\alpha_a, \alpha_b \in [6] \backslash L(V_0)$ such that $L(V_8) \cap L(V_2) = \{\alpha_a, \alpha_b\}$. Thus $L(V_8) = \{\alpha_a, \alpha_b, \alpha_c\}$ and $L(V_2) = \{\alpha_a, \alpha_b, \alpha_d\}$ such that $L(V_2) \cup L(V_8) = [6] \backslash L[V_2]$. Now by Proposition 3.10 (d) still, $L(V_4) \subseteq [6] \backslash L(V_2) = L(V_0) \cup \alpha_k$, such that $\alpha_k \notin L(V_0)$, $\alpha_k \in L(V_2)$. $L(V_0) \subseteq [6] \backslash L(V_8) = L(V_0) \cup \alpha_j$ for $\alpha_j \notin L(V_2)$, $\alpha_k \notin L(V_0)$, $\alpha_j \neq \alpha_k$. Thus, $|L(V_0) \cup \{\alpha_j \cup \alpha_k\}| = 5$. By $|L(V_6 \cup L(V_4))| = 6$, and for all $v_6 \in V_6$, and $v_4 \in V_4$, $d(v_4, v_6) = 2$. Thus a contradiction and hence $\lambda(G') \geq 7$.
Conversely, we consider the $7 - L(1,1)$-labeling of $C_{10} \times C_6$ in Figure 5 below.

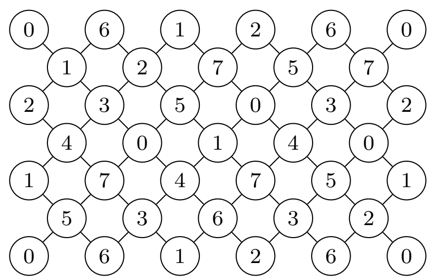

Fig. 7: $7 - L(1,1)$- labeling of $C_{10} \times C_6$

□

Theorem 3.16. $\lambda_1^1(C_{14} \times C_6) = 6$

Proof. Since 14 is not a multiple of 4 and by Corollary 3.13, $\lambda_1^1(C_{14} \times C_6) \geq 6$.

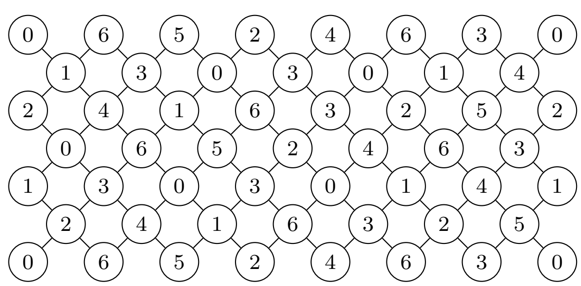

Fig. 8: $5 - L(1,1)$- labeling of $C_{14} \times C_6$

□

We can conclude that for $C_m \times C_6$, if m is even, and $m \not\equiv 0 \mod 4$, then $\lambda_1^1(C_m \times C_6) \geq 6$. For $m \geq 14$, this class of direct product graphs can be obtained from $C_{14+4m'}$, where m' is a non-negative integer.

Theorem 3.17. *For $m = 14+4m'$, where m' is a non-negative integer,* $\lambda_1^1(C_m \times C_6) = 6$

Proof. Since for any non-negative integer m', $m = 14 + 4m' \not\equiv 0 \mod 4$, then by 3.13, $\lambda_1^1(C_m \times C_n) \geq 6$. By combining the labeling in Figure 8 and $m'-$multiple of the labeling in Figure 3, we have that $\lambda_1^1(C_m \times C_n) \leq 6$ and the result follows. □

Note that the following result was established in [2]

Theorem 3.18. *Let $m', n' \equiv 0 mod 10$ and $A = \{12, 14, 16, 18\}$. Then, for all $k \in A and m, n$, $\lambda_1^1(C_{m'} \times C_{k+n'}) = 5$. Also, let $m, n \equiv 0 mod 5$, then $\lambda_1^1(C_m \times C_n) = 4$.*

4. Labeling of $C_m \times C_n,\ n \geq 8$

In this section, we obtain the λ_1^1-numbers of graph product $C_m \times C_n$, where $n, m \geq 8$.

Now we establish the λ_1^1-number for $C_m \times C_8$. Since labeling of product graphs is commutative, we restrict our work in this section to $m \geq 8$ since the cases for smaller graphs have been taken care of in the last sections.

The following result are helpful to reveal some useful properties of $L(1, 1)$- labeling of $C_m \times C_8$.

Lemma 4.1. *Let G' be a connected component of $C_m \times C_8,\ m \geq 4$. Suppose there exist $v_a, v_b \in V_i$ such that $\alpha_k = l(v_a) = l(v_b) \in [p]$, $p \in \mathbb{N}$. then $\alpha_k \notin L(V_{i+1} \cup V_{i+2})$. Furthermore, $\alpha_k \notin L(V_{i-1} \cup V_{i-2})$.*

Proof. Claim: Let $\alpha_k = l(v_a) = l(v_b)$, $v_a, v_b \in V_i$, $V_i \subseteq V(G')$. Then, $d(v_a, v_b) = 4$. Reason: Clearly, $|V_i| = 4$ for $i \in [m-1]$. Let $V_i = \{u_iv_0, u_iv_2, u_iv_4, u_iv_6\}$. So, $d(u_iv_0, u_iv_2)$=$d(u_iv_2, u_iv_4)$=$d(u_iv_4, u_iv_6)$=2. also, $d(u_iv_6, u_iv_4) = 2$ since C_8 is a cycle. However, $d(u_iv_0, u_iv_4)$=$d(u_iv_2, u_iv_6) = 4$. Thus $v_a = v_iv_0$ and $v_b = v_iv_4$ or $v_a = v_iv_2$ and $v_b = v_iv_6$. Now, suppose $v_a = u_iv_0$ and $v_b = u_iv_4$. Let $V_{i+1} = \{u_{i+1}v_1, u_{I+1}v_3, u_{i+1}v_5, u_{i+1}v_7\}$. Then that $d(u_iv_0, u_{i+1}v_1) = 1 = d(u_iv_0, u_{i+1}v_7)$ follows from the definition of $C_m \times C_{n=8}$. Likewise, $d(u_iv_4, u_{i+1}v_3) = 1 = d(u_iv_4, u_{i+1}v_5)$. Therefore, $\alpha_k \in L(V_{i+1})$. Also, let $V_{i+2} = \{u_{i+2}v_0, u_{i+2}v_2, u_{i+2}v_4, u_{i+2}v_6\}$. Then $d(u_iv_0, u_{i+2}v_0) = 2$=$d(u_iv_0, u_{i+2}v_2)$ and $d(u_iv_0, u_{i+2}v_6)$=2=$d(u_iv_0, u_{i+2}v_2)$ since $C_{m=8}$ is a cycle. Now $d(u_iv_4, u_{i+2}v_{2(4,6)}) = 2$ and therefore $\alpha_k \notin L(V_{i+2})$. This argument is valid for V_{i-1} and V_{1-2}. □

The consequence of Lemma 4.1 is that if a label is assigned to two vertices on $V_i \subset V(G')$, then the label could no longer be assigned to another vertex on the vertex sets two step above or below it. The next result is similar.

Proposition 4.2. *Suppose $v_i \in V_i$ and $v_{i+2} \in V_{i+1}$ such that $\alpha_k = l(v_i) = l(v_{i+1})$, then, $d(v_i, v_{i+1}) = 3$.*

Proof. Suppose $v_i = u_iv_0$ without loss of generality, then $d(v_i, u_{i+1}v_{1(7)}) = 2$. Now. $d(v_i, u_{i+1}v_3) = 3$ and $d(v_i, u_{i+1}v_5) = 3$. □

Lemma 4.3. *Suppose $V_1, V_{i+1} \subset V(G')$ where G' is a connected component of $C_m \times C_8$. Let $\alpha_k \in L(V_i) \cap L(V_{i+1})$ then $\alpha_k \notin L(V_{i-1}) \cup L(V_{i+2})$.*

Proof. By Proposition 4.2, suppose that $\alpha_k = l(u_iv_j)$ and that $\alpha_k \in L(V_{i+1})$ then, $\alpha_k = l(u_{i+1}v_{j+3})$ or $\alpha_k = l(u_{i+1}v_{j+5})$. Without loss of generality, suppose that in fact, $\alpha_k = l(u_{i+1}v_j{+}5)$. Then $d(u_iv_j, u_{i+2}v_{j(j+2,j+6)}) = 2$. Meanwhile, $d(u_{i+1}v_{j+3(j+5)}, u_{i+2}v_{j+4}) = 1$. Thus, $\alpha_k \notin L(V_{i+2})$. Similar argument holds for $\alpha_k \notin L(V_{i-1})$. □

By to Lemma 4.3, it is quite clear that if α_k belongs $L(V_i) \cap L(V_i + 1)$, then α_k does not belong to $L(V_{i-1} \cup Vi + 2)$. A similar result is as follows:

Lemma 4.4. *Suppose $\alpha_k \in L(V_i) \cap L(V_{i+2}) \subset V(G')$, where G' is a connected component of $C_m \times C_8$, $m \geq 4$, then $\alpha_k \notin L(V_{i+1})$*

Proof. Let, $v_i \in V_i$ be u_iv_0. Note that, $d(u_iv_0, u_{i+2}v_{0(2,6)}) = 2$ since C_8 is a cycle. Then, the remaining vertex $v_{i+2} \in V_{i+2}$ such that $l(v_{i+2}) = \alpha_k$ is $u_{i+2}v_4$ and $d(u_iv_0, u_{i+2}v_4) = 4$. Now $d(v_i, u_{i+1}v_{1(7)}) = 1$ and $d(u_{i+2}v_4, u_{i+1}v_{3(5)}) = 1$. Thus, $\alpha_k \notin L(V_{i+1})$. □

The consequence of Lemma 4.4 is that if two vertices on V_i and V_{i+2} share the same label, then that label can not be shared by another vertex on V_{i+1} given that V_i, V_{i+1} and V_{i+2} are all in $V(G')$.

Next we establish the lower bound of $\lambda_1^1(C_m \times C_8)$ where $m \geq 8$ and $m \equiv 2 \mod 6$. We require the following definition.

Let G' be a connected component of G. Then, V_{α_k} is the class of all vertices on $V(G')$ labeled α_k.

Lemma 4.5. *For $m \geq 8, m \equiv 2$ mod 6, $\lambda_1^1(C_m \times C_8) \geq 6$.*

Proof. Case 1: Let $\alpha_k \in L(V(G'))$ such that if $\alpha_k \in L(V_i)$, $V_i \subset V(G')$, $i \in [m-1]$, then there exist $v_i', v_i'' \in V_i$ such that $l(v_i') = \alpha_k = l(v_i'')$. Let $\bar{V}$ be a class of all $V_i \in V(G')$ such that $\alpha_k \in L(V_i)$. Now suppose, without loss of generality, that $V_0 \in \bar{V}$. By this and Lemma 4.1, and by assuming that α_k labels $V(G')$ optimally, suppose $V_i \in \bar{V}$, then $i \equiv 0 \mod 3, i \neq m-2$. Since $m \equiv 2 \mod 6$, then there exists $n' \in \mathbb{N}$, such that $m = 6n' + 2$. Note that $m - 5 = (6n' + 2) - 5 = 3(2n' - 1)$. Thus, $V_{m-5} \in \bar{V}$. By Lemma 4.1 and since $V_0, V_{m-5} \in \bar{V}$, then $\alpha_k \notin L(V_{m-4} \cup V_{m-3} \cup V_{m-2} \cup V_{m-1})$. Thus $\bar{V} = \{V_0, \cdots, V_{m-5}\}$. Set $\bar{V}' = \bar{V} \setminus \{V_0\}$. Since $\left|\bar{V}'\right| = 2n' - 1$, then $|V'| = 2n'$. Now, $|V(G')| = (6n' + 2)4$. Clearly $|V_{\alpha_k}| = 2(2n') = 4n'$. Hence, $\frac{|V(G')|}{\left|V_{\alpha_k}\right|} = \frac{6n'+2}{n'} > 6$.

Case 2: Suppose that for all triple $V_i, V_{i+1}V_{i+2} \subset V(G)$, $\alpha_k \in L(V_i \cap V_{i+1})$ and by Lemma 4.3 $\alpha_k \notin V_{i+2}$. Without loss of generality, we select the initial triple to be V_0, V_1, V_2, such that $\alpha_k \in L(V_0 \cap V_1)$, $\alpha_k \notin V_2$; (and $\alpha_k \in L(V_3 \cap V_4)$,$\alpha_k \notin L(V_5) \cdots$). Therefore, $\alpha_k \notin V_i$ for all $i \in [m-1]$ such that $i + 1 \equiv 0 \mod 3$. Now, $m \equiv 2 \mod 6$ implies there exists $n' \in \mathbb{N}$ such that $m = 6n' + 2$. Thus, $m - 2 \equiv 0 \mod 3$ and hence $\alpha_k \notin L(V_{m-3})$. Now, since $\alpha_k \in L(V_0) \cap L(V_1)$, then $\alpha_k \notin L(V_{m-1})$ by Lemma 4.3 and since C_m is a cycle. By Lemma 4.4, it is possible for $\alpha_k \in L(V_{m-2})$ since $\alpha_k \notin L(V_{m-3})$. Thus we, for maximality, assume that $\alpha_k \in L(V_{m-2})$. Now, Let $\bar{V_{\alpha_k}} = \{V_0, \cdots, V_{m-3}\} \subset V(G')$. Then $\left|\bar{V_{\alpha_k}}\right| = \left\lceil \frac{(m-3)+1}{3} \right\rceil = 2(\frac{6n'}{3}) = 4n'$, where $n' \in \mathbb{N}$. Thus, for all $V_i \in V(G')$, $|V_{\alpha_k}| = 4n' + 1$ since $\alpha_k \in L(V_{m-2})$. Thus $\frac{|V(G')|}{\left|V_{\alpha_k}\right|} = \frac{(6n'+2)4}{4n'+1} > 6$.

Case 3. Suppose that, by Lemma 4.4, $\alpha_k \in L(V_i, V_{i+2}, V_{i+4}, \cdots, V_{i-2})$. Clearly, $|V_{\alpha_k}| = \frac{m}{2}$, since m is even. Now, $m = 2 \mod 6$ implies that there exists $n' \in \mathbb{N}$ such that $m = 6n'+2$. Therefore $|V_{\alpha_k}| = 3n'+1$. Now $\frac{|V(G')|}{|V_{\alpha_k}|} = \frac{(6n'+2)4}{3n'+1} > 8$. It is easy therefore to see that combination of the Cases 1-3 will still result in $\frac{|V(G')|}{V_{\alpha_k}} \geq 7$. Thus for all $\alpha_k \in [p]$, where $\alpha_k \in [p]$, $\lambda_1^1(G') = p, \frac{|V(G')|}{|V_{\alpha_k}|} \geq 7$. Suppose $\lambda_1^1(G') = p = 5$ and the maximum number of vertices in G' that $\alpha_k \in [p]$ labels for all $\alpha_k \in [p]$ is V_{α_k}, then $(p+1)V_{\alpha_k} \geq |V(G')|$ implies that $p+1 \geq \frac{|V(G')|}{|V_{\alpha_k}|}$. This implies that $\frac{|V(G')|}{|V_{\alpha_k}|} \leq p+1$. Now, since $p = 5$, then $\frac{|V(G')|}{|V_{\alpha_k}|} \leq 6$, which is a contradiction since in fact, $\frac{|V(G')|}{|V_{\alpha_k}|} \geq 7$. Thus $\lambda_1^1(C_m \times C_8) \geq 6$ for all $m \equiv 2 \mod 6$. □

Next, we consider the second case of $m \equiv 4 \mod 6$.

Lemma 4.6. *For $m \equiv 4$ mod 6, $\lambda_1^1(C_m \times C_8) \geq 6$.*

Proof. Case 1: Let G' be a connected component of $C_m \times C_8$, $m \equiv 4 \mod 6$ and let $\bar{V}$ be a set of $V_i \subset V(G')$ such that for all i there exist $v_i', v_i'' \in V_i$ such that $l(v_i') = \alpha_k = l(v_i'')$. Now suppose $V_0 \in \bar{V}$. By Lemma 4.1, $\alpha_k \notin L(V_1 \cup V_2)$. Since $\bar{V}$, contains all possible $V_i \subset V(G)$ and since $V_0 \in \bar{V}$, then for all $i \equiv 0 \mod 3$, $V_i \in \bar{V}$ except for $i = m-1$ since C_m is a cycle and $V_0 \in \bar{V}$. We know that $m = 6n'+4$, $n' \in \mathbb{N}$ and thus, $m-4 = 0 \mod 3$, which implies that $V_{m-4} \in \bar{V}$. Set $\bar{V}' = \{V_3, \cdots, V_{m-4}\}$. Thus, $|V'| = \frac{m-4}{3} = \frac{6n'+4-4}{3} = 2n'$. Now, $\bar{V} = \bar{V}' \cup V_0$. Thus $\left|\bar{V}\right| 2n'+1$ and $|V_{\alpha_k}| = 2(2n'+1) = 4n'+2$. Now, $|V(G')| = 4(6n'+4) = 24n'+16$. Finally, $\frac{|V(G)|}{|V_{\alpha_k}|} = \frac{24n'+16}{4n'+2} > 6$

Case 2: Suppose that for all triple $V_i, V_{i+1}, V_{i+2} \subset V(G)$, $\alpha_k \in L(V_i) \cap L(V_{i+1})$ and $\alpha_k \notin L(V_{i+2})$. We can select the initial triple as V_0, V_1, V_2, that is, $\alpha_k \in L(V_0) \cap L(V_1)$ and $\alpha_k \notin L(V_2)$ (and subsequently, $\alpha_k \in L(V_3) \cap L(V_4)$ and $\alpha_k \notin L(V_5) \cdots$). Thus, $\alpha_k \notin V_i$ for all i such that $i+1 \equiv 0 \mod 3$. Now since $m \equiv 4 \mod 6$ there exists $n' \in \mathbb{N}$ such that $m \equiv 6n'+4$. Clearly, $m-1 = 6n'+3 = 3(2n'+1) \equiv 0 \mod 3$. However, $\alpha_k \notin L(V_{m-1})$ since C_m is a cycle and by the Lemma 4.3. Therefore let $\bar{V} = \{V_0, V_1, V_2, \cdots, V_{m-2}\} \subseteq V(G')$. Then $\left|\bar{V}\right| = m-2+1 = m-1$. Clearly $|V_{\alpha_k}| = 2\frac{\left|\bar{V}\right|}{3} = \frac{2(m-1)}{3}$. The last equation implies that $\frac{2(6n'+4-1)}{3} = \frac{2\cdot 3(2n'+1)}{3} = 2(n'+1)$, $n' \in \mathbb{N}$. Now, $|V(G')| = 4m = 4(6n'+16) = 24n'+16$. Therefore, $\frac{|V(G')|}{|V_{\alpha_k}|} = \frac{24n'+16}{4n'+2} > 6$.

Case 3: This follows similar argument as in Case 3, in the proof of Lemma 4.5.

Therefore for m even, $m \equiv 4 \mod 6$, $\lambda_1^1(G') \geq 6$ follows similar argument as in proof of Lemma 4.5. □

Corollary 4.7. *For all m even, $m \not\equiv 0 mod 6$, $\lambda_1^1(C_m \times C_8) \geq 6$.*

Proof. It follows from combining the results in Lemmas 4.5 and 4.6. □

Next we obtain the λ_1^1 number of a special case of Corollary 4.7.

Theorem 4.8. $\lambda_1^1(C_8 \times C_8) = 7$

Proof. By following the the process in the proof of Lemma 4.5, we have that $|V_{\alpha_k}| = 4$, for $V_{\alpha_k} \subseteq V(G')$, where G' is a connected component of $C_8 \times C_8$ and therefore $\frac{|V(G')|}{|V_{\alpha_k}|} = \frac{32}{4} = 8$. Thus $\lambda_1^1(C_8 \times C_8) \geq 7$. From an earlier result, $\lambda_1^1(C_4 \times C_8) = 7$. By copying re-occurrence of the labeling of $C_4 \times C_8$, we have that $\lambda_1^1(C_8 \times C_8) \leq 7$ and the result follows. □

In what follows, we extend our result to $m \geq 10$.

Theorem 4.9. *Let* $m \in \{10, 14\}$. *Then* $\lambda_1^1(C_m \times C_8) = 6$

Proof. By Corollary 4.7, $\lambda_1^1(C_m \times C_8) \geq 6$ for all $m \in \{10, 14\}$. Conversely, we show that for $m \in \{10, 14\}$, $\lambda_1^1(C_m \times C_8) \leq 6$ by labeling their connected component as shown below.

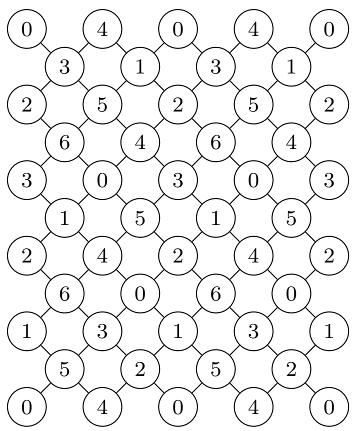

Fig. 9: $6 - L(1,1)$-Labeling of $C_{10} \times C_8$

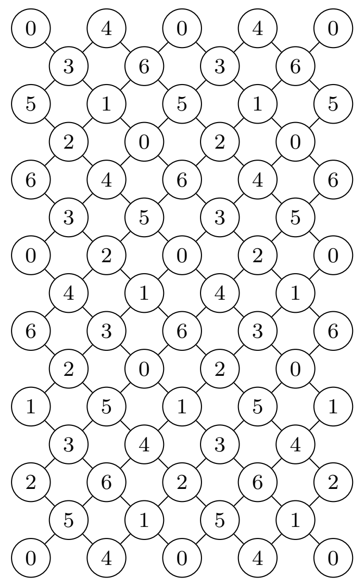

Fig. 10: $6 - L(1,1)$-Labeling of $C_{14} \times C_8$

□

In the next result we show that for all $m \equiv 2 \mod 6$ and $m \equiv 4 \mod 6$, $m \geq 14$, $\lambda_1^1(C_m \times C_8) = 6$

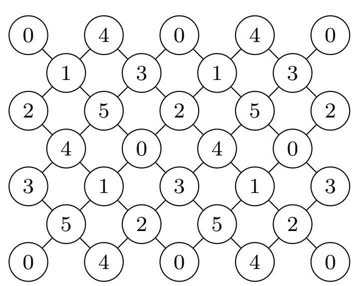

Fig. 11: $5 - L(1,1)$-Labeling of $C_6 \times C_8$

Theorem 4.10. *Let $m \not\equiv 0 \mod 6$, $m \geq 10$ and even. Then $\lambda_1^1(C_m \times C_8) = 6$.*

Proof. By Corollary 4.7, we see that for $m \not\equiv 0 \mod 6$, $m \geq 16$, $\lambda_1^1(C_m \times C_8) \geq 6$. Now, by combining the $m'-$copies of the labeling in Figure 1, with the $n'-$copies of labeling in Figure 11, $m', n' \in \mathbb{N}$ we have that $\lambda_1^1(C_{10m'+6n'} \times C_8) \leq 6$, with $10m' + 6n' \equiv 4 \mod 6$. By combining the labeling in Figure 10 with the $n'-$copies labeling Figure 11, $n' \geq 1$, $n' \in \mathbb{N}$ we have that $\lambda_1^1(C_{14+6n'} \times C_8) \leq 6$, with $14+6n' \equiv 4 \mod 6$. Thus, $C_m \times C_8 \leq 6$ for all $m \geq 16, m \not\equiv 0 \mod 6$. Note that if $n', m' = 0$, then we have the $L(1,1)$-labeling of $C_m \times C_8$, where $m \in \{10, 14\}$, which are done in Theorem 4.9 □

In what comes next, we obtain the $\lambda_1^1(C_m \times C_{10})$. Our result will be based on that of $P_m \times C_n$.

Lemma 4.11. *For all $m \geq 9$, $n \geq 12$, $\lambda_1^1(C_m \times C_n) \geq 5$.*

Proof. It is easy to see that $P_m \times C_n \subseteq C_m \times C_n$. Therefore the claim follows from Lemma 2.4. □

Now that the lower bound has been shown for $C_m \times C_n$, for specific lengths of cycles, we proceed to establish the optimal $L(1,1)$-numbers for various graphs in this class. In the case of $C_m \times C_{10}$, see Theorem 3.18.

Theorem 4.12. *For $m \geq 3$, $\lambda_1^1(C_m \times C_{12}) = 5$*

Proof. For all $m \equiv 0 \mod 4$, or $m \equiv 0 \mod 6$, and by commutativity of $C_m \times C_n$ the claim follows from Corollary 3.5. We now need to show the result for $m \not\equiv 0 \mod 4$, $m \not\equiv 0 \mod 6$. It is easy to see that such number, m', is obtainable from this formula: $m' = p + 2$, where $p \in \mathbb{N}$, $p \equiv 0 \mod 4, 0 \mod 6$. The first of such number is 14. We need a $5-$labeling of $C_{14} \times C_{12}$.

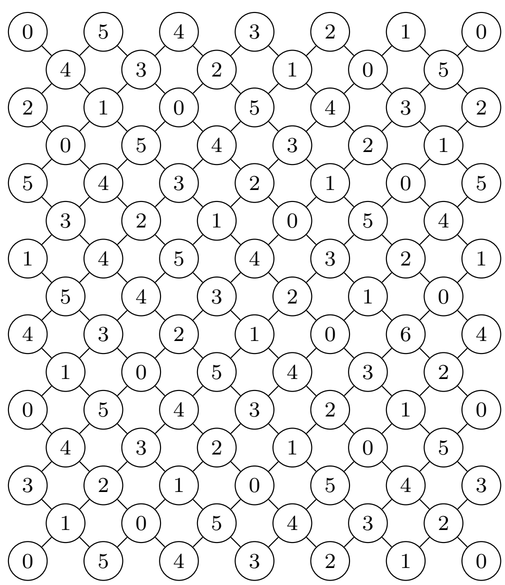

Fig. 12: $5 - L(1,1)$-Labeling of $C_{14} \times C_{12}$

Note that $\cup_{i=0}^{4}\{V_i\}$ in Figure 12 above forms a component of a $C_4 \times C_{12}$ and $L(\cup_{i=0}^{4}\{V_i\}) = [5]$. Therefore there exist an independent $5 - L(1,1)$-labeling of $C_4 \times C_{12}$ in the $L(1,1)$-labeling of $C_{14} \times C_{12}$. Thus $\lambda_1^1(C_{10+4m'} \times C_{12}) = 5$, $m' \in \mathbb{N}$. Since for all $p \in \mathbb{N}$, p can be expressed as $10 + 4m'$, then the required result holds. □

Corollary 4.13. *For all $m \equiv 0$ mod 14, $n \equiv 0$ mod 12, $m, n \not\equiv 0$ mod 5, $\lambda_1^1(C_m \times C_n) = 5$.*

The next result establishes an optimal $L(1,1)$-labeling of $C_m \times C_n$ of a certain size. This resolves all cases of large enough m and n.

Theorem 4.14. *For $m', m'', n', n'' \in \mathbb{Z}_+$ $\lambda_1^1(C_{10m'+14m''} \times C_{10n'+12n''}) = 5$.*

Proof. From earlier results, $\lambda_1^1(C_m \times C_n) \geq 5$ for $C_m \times C_n$ defined in the statement above. Each of the quadrant in Figure 13 represents special $5 - L(1,1)$-labelings of $C_{10} \times C_{10}$, $C_{10} \times C_{12}$, $C_{10} \times C_{14}$ and $C_{12} \times C_{14}$ respectively. Clearly, these labelings form a $5 - L(1,1)$-labeling of $C_{10+14} \times C_{10+12}$. Thus, for $m', m'', n', n'' \in \mathbb{Z}_+$, $\lambda_1^1(C_{10m'+14m''} \times C_{10n'+12n''}) \leq 5$.

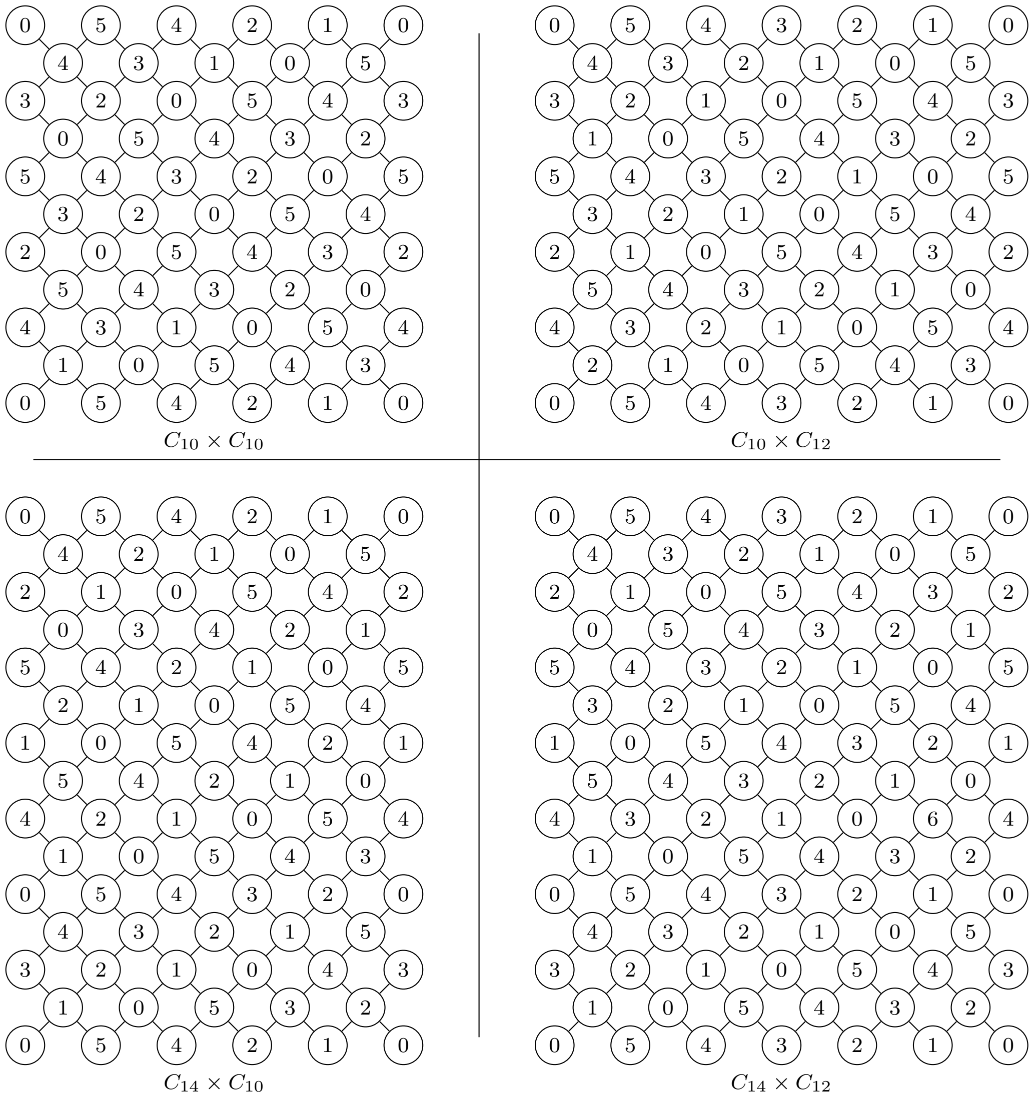

Fig. 13: $5-L(1,1)-$labeling of $C_{10m'+14m''} \times C_{10n'+14n''}$, for all $m', m'', n', n'' \in \{0,1\}$.

□

Corollary 4.15. *For $m \geq 48$ and $n \geq 40$, $\lambda_1^1(C_m \times C_n) = 5$.*

The last corollary gave the values of m, n beyond which $\lambda_1^1(C_m \times C_n) = 5$. However, there are smaller product graphs whose $L(1,1)$-number is 5 as demonstrated in the next corollary.

Corollary 4.16. *For all $m, n \geq 14$, $m \notin \{14, 16, 18, 22, 26, 32, 36, 46\}$, $n \notin \{14, 16, 18, 26, 28, 34\}$, $\lambda_1^1(C_m \times C_n) = 5$.*

For some of pairs $\{m', n'\}$, in the two sets defined above, namely $\{16, 18\}$, $\{18, 28\}$,$\{32, 18\}$,$\{36, 16\}$,$\{36, 18\}$,$\{36, 28\}$, $\lambda_1^1(C_m \times C_n) = 5$. This is obvious from earlier results. For the remaining pairs, it can easily be confirmed, by manual labeling,

that $\lambda_1^1(C_m \times C_n) \leq 6$. However, we observe that there could be a better upper bound and therefore, present the following conjecture:

Conjecture 4.17. *For $m, n \geq 12$, $m, n \not\equiv 0 \mod 5$, $\lambda_1^1(C_m \times C_n) = 5$.*

Solving this conjecture only requires confirming the $5-L(1,1)$-labeling for $C_m \times C_n$, where m, n are the remaining pair yet to be confirmed in the sets in Corollary 4.16.

The results obtained is summarized in the table below:

m	n	$\lambda_1^1(C_m \times C_n)$
4	$4, 5, 8, 10$	7
4	$n \not\equiv 0 \mod 3$ & $n \geq 11$	6
$m \equiv 0 \mod 4$	$n \equiv 0 \mod 3$	5
$m \equiv 0 \mod 5$	$n \equiv 0 \mod 5$	4
6	$3, 6$	8
6	$5, 10$	7
6	$7, 9, 11, 14 + 4n', n' \geq 0$	6
8	8	7
8	$n \geq 10, n \not\equiv 0 \mod 3$	6
$m \equiv 0 \mod 10$	$n \geq 11, n \not\equiv 0 \mod 5$	5
12	$n \geq 12$	5
$m \geq 40, even$	$n \geq 48$	5

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